The Area of the Solid of Intersection of a Sphere and an Ellipsoid, a First Approach

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In this paper we consider curves of intersection of a fixed ellipsoid and members of a family of spheres with common center and different radii. We use Maple to obtain the exact intersection curve using rectangular coordinates. We then determine the surface area of the portion of the ellipsoid inside the sphere and vice versa. Along the way we discuss several issues of interest to students and instructors of Calculus. We provide examples to illustrate the various possibilities that arise and we provide Maple worksheets that can be used to deal with the rather complicated calculations that must be performed. These worksheets were used with version 14 of Maple to obtain the results reported in this paper.

1 Introduction

The problem considered in this paper is motivated by amusing and well-known two-dimensional problems [3]. In one version of the basic problem a goat is tethered by a rope to a tree located at the center of a circle. The tree lies outside an elliptically shaped park. The objective is to find the length of the rope for which the area of the portion the goat has eaten is equal to the area of the uneaten portion. If this problem is extended to three-dimensions, the objective becomes that of finding the radius of a sphere such that the surface areas of the portions of the sphere and an ellipsoid that are inside one another are equal.

As it turns out, the question addressed in this paper is applicable in more important contexts than the above problem. In fact, the intersection of two quadrics is a question that has been studied extensively in various contexts because of its usefulness in computer aided design, solid modeling, and design of mechanical parts (see [1] and the references therein for detailed discussions of the relevant issues). We limit our attention to the question for ellipsoids and spheres. Without loss of generality, we assume the ellipsoid is fixed and centered at the origin with the equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1 \tag{1}$$

where a, b, c are fixed positive constants.

We denote by (h, k, l) the fixed center of the sphere and by r the radius of the sphere. The equation of the sphere is then

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2 \tag{2}$$

We wish to find the exact intersection curve and then perform surface area calculations. Despite difficulties that we will identify for this approach, we use rectangular coordinates in our approach both because this approach is of interest in its own right and because it provides independent verification of the results for a second approach based on parametric equations and developed in [5]. We will see that Maple [2] supplemented by capable numerical procedures is able to perform the seemingly intractable calculations inherent in our approach.

We note that the intersectplot command can be used to plot the intersection curve as follows:

```
Surface1 := (x/a)^2 + (y/b)^2 + (z/c)^2 = 1:
Surface2 := (x-h)^2 + (y-k)^2 + (z-l)^2 = r^2:
Both := intersectplot(Surface1,Surface2,x=-a..a,y=-b..b,z=-c..c,
thickness=5,color=yellow,scaling=constrained,axes=boxed):
```

We should point out that the abbreviated code snippets that are used in this paper to illustrate basic calculations will not necessarily run as stated. For example, the above code will execute only after the plots package has been loaded and numerical values for a, b, c, h, k, l have been prescribed. Refer to (1) in §7 for the detailed commands for the illustrative code snippets given in this paper.

We will obtain the curve in a more complicated manner described in the next section. We do this in order to obtain a precise description of the curve and to facilitate the calculation of the relevant surface areas.

2 Solution Procedure

If (1) and (2) are solved for z and the results are equated, any of the four possible resulting equations may be used to solve for y using the Maple solve command. The curve $(x, y(x), 0)$ is the projection of the intersection curve onto the xy-plane. A straightforward selection procedure can then be used to find the points $(x, y(x), z(x, y(x)))$ on the intersection curve.

The solution $y(x)$ is complicated. Embedded in each branch are zeroes of a quartic polynomial. These zeroes generally define four solution branches. (In some special cases discussed in §3 there are fewer than four branches.) Specifically for the non-faint of heart, the branches $y_i(x)$ are given by the roots of a quartic polynomial:

```
RootOf(
+(c^4+b^4-2 b^2 c^2) Z^4 + (-4 a k b^3+4 a c^2 k b) Z^3
+(6 a^2 b^2 k^2+2 a^2 b^2 h^2+2 a^2 b^2 l^2-4 a^2 b^2 x h
+2 a^2 b^2 c^2-2 a^2 b^2 r^2-2 x^2 b^2 c^2+2 a^2 b^2 x^2
-2 a^2 c^4-2 c^2 a^2 k^2+2 c^4 x^2+4 a^2 c^2 x h
+2 a^2 r^2 c^2+2 a^2 c^2 l^2-2 a^2 c^2 h^2-2 a^2 x^2 c^2) Z^2
+(-4 b a^3 x^2 k-4 b a^3 l^2 k+8 b a^3 x h k-4 b a^3 h^2 k
```

```
-4 b a^3 c^2 k+4 b a^3 r^2 k-4 b a^3 k^3+4 b x^2 c^2 k a) Z
+(2 a^4 c^2 h^2-2 x^4 c^2 a^2+2 a^4 c^2 k^2-2 a^4 c^2 l^2
+2 a^4 c^2 x^2+6 x^2 h^2 a^4+2 k^2 h^2 a^4-2 h^2 r^2 a^4
-2 r^2 l^2 a^4-4 x h^3 a^4+l^4 a^4+a^4 c^4+x^4 c^4
+k^4 a^4+h^4 a^4+r^4 a^4+x^4 a^4-2 a^4 r^2 c^2-2 a^2 c^4 x^2
-2 x^2 r^2 a^4-2 k^2 r^2 a^4+2 x^2 l^2 a^4+2 h^2 l^2 a^4
+2 k^2 l^2 a^4-4 h x^3 a^4+2 k^2 x^2 a^4+4 h x r^2 a^4
+2 x^2 c^2 a^2 l^2-2 x^2 c^2 a^2 k^2+2 a^2 r^2 x^2 c^2
-4 h x l^2 a^4+4 x^3 c^2 a^2 h-4 a^4 c^2 x h
-2 x^2 c^2 a^2 h^2-4 k^2 h x a^4) ) b/a
```

Obtaining the general roots of the above expression is too time consuming to be feasible. It requires several hours of execution time and the general roots produced are each several hundred thousand lines long. However, after assigning values for the constants a, b, c, h, k, l, r, the roots can be obtained easily and quickly in the following manner:

```
ZEP := (x,y) -> c * sqrt(1 - (x/a)^2 - (y/b)^2):
ZSP := (x,y) -> l + r * sqrt(1 - ((x-h)/r)^2 - ((y-k)/r)^2):
# Use either the top or bottom for each quadric.
  eqnSE := ZSP(x,y) = ZEP(x,y):
# This produces the above RootOf:
  yvalSE := solve(eqnSE,y);
# The solution branches are then:
# yvalSE[1]
# yvalSE[2]
# yvalSE[3]
# yvalSE[4]
```

We have successfully used version 14 of Maple to perform these calculations. On a Dell Inspiron laptop, the necessary calculations typically require 5-15 seconds depending on the complexity of the intersection of the two quadrics. (1) in §7 can be used to perform the relevant calculations.

The real domain for each of the four $y(x)$ branches must be determined. It is necessary to determine these domains in order to set up the surface area integrals of interest. Each branch is real for some values of x and complex for others. We use root finding to determine the real domain for each branch and the points at which the branches connect. For this root finding we use Zeromw.mws, a Maple adaptation of a well-known root finder from [4]. The details of the various root finding done are documented in (2) in §7. Often the domains consist of more than one interval (see Example 3). In this case each interval is determined recursively.

In order to perform the desired surface area calculations, we proceed as follows. Each of the integrals evaluated has the form

$$\int_{x_1}^{x_2} \int_{\phi_1(x)}^{\phi_2(x)} \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\; dy\, dx \tag{3}$$

where $f(x, y)$ is obtained from (1) and 2. The outer limits x_1 and x_2 and the inner limits $\phi_1(x)$ and $\phi_2(x)$ correspond to closed regions on the equator planes for the two quadrics (that is, the plane $z = 0$ for the ellipsoid and the plane $z = l$ for the sphere). $\phi_1(x)$ and $\phi_2(x)$ consist either of portions of the $y(x)$ solution branches that the surface region bounded by the intersection curve projects onto the relevant equator plane (along with portions of the circle or ellipse in the equator plane in situations for which the projected solution branches intersect the circle or ellipse).

After the intersection curve has been located, the portions of the curve above and below the ellipsoid's equator are projected onto the plane $z = 0$ to determine the relevant closed region on the equator plane. Similarly, the portions of the curve above and below the sphere's equator are projected onto the plane $z = l$. In the first case, the projected domains along with the ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1;\ z = 0$$

determine two-dimensional domains corresponding to points above or below the plane $z = 0$. Throughout this paper and in the figure captions we will refer to these projections of the intersection curve as spherical and ellipsoidal projections.

The ellipsoidal inner integral for each surface area can be determined analytically (which involves using elliptic integrals). In the second case, the projected domains along with the circle

$$(x - h)^2 + (y - k)^2 = r^2;\ z = l$$

determine two-dimensional domains corresponding to points above or below the plane $z = l$. As in the first case the inner integral can be determined analytically (see below). One-dimensional integrations are then performed to determine the surface area for each of the pieces. Adaptmw.mws, a Maple adaptation of a well-known Gauss-Konrod algorithm from [4], is used to perform the integrations.

Each of the four integrations can, in fact, require several integrations depending on the manner in which the $y(x)$ branches join and whether the $y(x)$ curves intersect the projected ellipse or sphere. (1) in §7 generates the necessary integration limits as well as several plots showing the colored branches. This allows the necessary integrals to be set up in a straightforward manner. The inner antiderivatives can be determined as follows. The incomplete elliptic integral produced for the ellipsoid can be evaluated using the Maple EllipticE command.

```
# Surface area antiderivative for the sphere ...
diff(ZSM(x,y),x);
diff(ZSM(x,y),y);
1+(diff(ZSM(x,y),x))^2+(diff(ZSM(x,y),y))^2;
sqrt(1+(diff(ZSM(x,y),x))^2+(diff(ZSM(x,y),y))^2);
simplify(%);
int(r/(r^2-(x-h)^2-(y-k)^2)^(1/2),y);
# The above produces the following antiderivative:
InnerS := (x,y) -> evalf(r*arctan((y-k)/(r^2-x^2
+2*x*h-h^2-y^2+2*y*k-k^2)^(1/2)));
```

```
# Surface area integrand for the ellipsoid ...
diff(ZEP(x,y),x);
diff(ZEP(x,y),y);
1+(diff(ZEP(x,y),x))^2+(diff(ZEP(x,y),y))^2;
sqrt(1+(diff(ZEP(x,y),x))^2+(diff(ZEP(x,y),y))^2);
simplify(%);
int((1/a^2/b^2 (a^4 b^4-a^2 b^4 x^2-a^4 b^2 y^2
+c^2 b^4 x^2+c^2 a^4 y^2)
/(a^2 b^2-x^2 b^2-y^2 a^2))^(1/2),y);
# The above produces the following antiderivative:
InnerE := (x,y) -> evalf(-(a^4-x^2 a^2+c^2 x^2)
EllipticE(y (a^2/b^2/(a^2-x^2))^(1/2),
((a^2 b^2-x^2 b^2-c^2 a^2+c^2 x^2) a^2/b^2
/(a^4-x^2 a^2+c^2 x^2))^(1/2)) (-(-a^4 b^4+a^2 b^4 x^2
+a^4 b^2 y^2-c^2 b^4 x^2-c^2 a^4 y^2)
/(a^4-x^2 a^2+c^2 x^2)/b^4)^(1/2) (-(-a^2 b^2+x^2 b^2
+y^2 a^2)/b^2/(a^2-x^2))^(1/2) b^4 (-a^2 b^2+x^2 b^2+y^2 a^2)
((-a^4 b^4+a^2 b^4 x^2+a^4 b^2 y^2-c^2 b^4 x^2-c^2 a^4 y^2)
/a^2/b^2/(-a^2 b^2+x^2 b^2+y^2 a^2))^(1/2)/(a^6 y^4 b^2
-a^6 y^4 c^2-2 y^2 a^6 b^4+2 y^2 a^4 b^4 x^2
-y^2 a^2 c^2 b^4 x^2+y^2 c^2 a^6 b^2-y^2 c^2 a^4 x^2 b^2
+b^6 a^6-2 b^6 a^4 x^2
+b^6 x^4 a^2+b^6 c^2 x^2 a^2-b^6 c^2 x^4)^(1/2)/
(a^2/b^2/(a^2-x^2))^(1/2)/((-a^4 b^4+a^2 b^4 x^2+a^4 b^2 y^2
-c^2 b^4 x^2-c^2 a^4 y^2) (-a^2 b^2+x^2 b^2+y^2 a^2))^(1/2));
```

3 Special Cases

Several special cases are noteworthy. For each of these special cases, if the relevant substitutions are made in the equations to be solved, it is a simple matter to verify the assertions in this section.

The intersection curve does not lie completely in one plane unless the ellipsoid is itself a sphere. In this case and only this case the intersection curve is, in fact, a circle. It is a simple matter to find the intersection curve as well as the surface areas in this case. (5) in §7 can be used to verify these remarks for the sphere-sphere case.

For the ellipsoid-sphere case remarks are in order if the center of the sphere lies on a coordinate axis. This poses no difficulty if the center lies on the z-axis (see Example 1). However, if the center lies on either the x- or y-axis, a “skinny” domain can result that can pose problems for the numerical solutions for (1) in §7. Fortunately, it is straightforward to show that the domain endpoints correspond to certain points of intersection of the ellipse and circle and certain other ellipses. Using this fact and symmetry, it is easy to set up the necessary surface area integrals. (3) and (4) in §7 contain the simplified solutions for these cases.

4 Illustrative Examples

We now present three examples that illustrate the salient features of the intersection curve of an ellipsoid and a sphere. Figures 1, 2, and 7 in this section were produced using the GinMA interactive graphics package described in [5]. The remaining figures were produced using Maple.

4.1 Example 1

For our first example, we use $(a, b, c) = (2, 3, 4)$, $(h, k, l) = (0, 0, 5)$ and $r = 2$. In this simple case the center of the sphere is on the z-axis. The intersection curve is determined by two real branches $y_1(x)$ and $y_2(x)$ where

$$y_1(x) = \frac{3}{7}\left(\sqrt{-541 - 21x^2 + 20\sqrt{760 + 35x^2}}\right).$$

and $y_2(x) = -y_1(x)$. (For the other two complex branches the coefficient 20 changes to -20.)

The real domain of $y_1(x)$ and $y_2(x)$ is approximately $(-1.1055, 1.1055)$. The intersection curve is below the equator of the sphere and above the equator of the ellipsoid. The curve intersects neither equator for the quadrics. Hence, one integral is required to evaluate each of the surface areas of the region of each quadric that is inside the other. The integration limits for the inner iterated integral are $y_1(x)$ and $y_2(x)$ in each case. The integration limits for the outer iterated integral are -1.1055 and 1.1055 in each case. The respective spherical and ellipsoidal surface areas are 5.4 and 5.8. These surfaces are depicted in Figure 1.

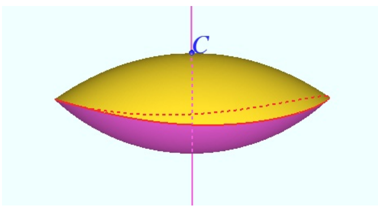


Figure 1: Enclosed Quadric Regions for Example 1

4.2 Example 2

For our second example, we use $(a, b, c) = (2, 3, 4)$, $(h, k, l) = (1, 2, 3)$ and $r = 2.2574$. The surface areas of the portion of the sphere inside the ellipsoid and the portion of the ellipsoid inside the sphere are each approximately equal to 13.8. The corresponding surfaces are depicted in Figure 2. Figure 3 depicts the curve of intersection of the sphere and ellipsoid. Two branches define the curve (the other two are complex). Figure 4 depicts the projection of the curve onto the xy-plane and the actual three-dimensional curve. Figure 5 shows the projection of the portion of the curve above $z = 0$ onto the plane $z = 0$. (The curve does not intersect the lower half of the ellipsoid.) Figure 6 shows the projections of the upper and lower halves of the curve onto the plane $z = l$. Inspection of the colored projected branches in these figures shows that five integrals are required to evaluate the surface area integrals (since the portion of the curve below $z = l$ requires three integrals).

It is instructive to consider the manner in which the integrals are defined for this example. First refer to Figure 6(a). The real domain for solution branches 1 and 2 is approximately $[-1.160, 1.825]$. For the portion of the intersection curve above $z = l$, the inner integration limits are the bottom of the circle $y = k - \sqrt{r^2 - (x-h)^2}$ and $y = y_2(x)$. The outer limits are $x_1 = -1.085$ and $x_2 = 1.314$. x_1 and x_2 are the points at which the intersection curve intersects the equator of the sphere. Next refer to Figure 6(b). For the portion of the intersection curve below $z = l$, three integrals are required, one each for the left and right "ears" and one for the middle region. We note that the analytic solution branches are extremely long for this example and will not reproduced here. For the left ear the inner limits are $y_1(x)$ and $y_2(x)$ and the outer limits are $x_1 = -1.160$ and $x_2 = -1.085$. For the middle region the inner limits are $y = k - \sqrt{r^2 - (x-h)^2}$ and $y = y_2(x)$. The outer limits are $x_1 = -1.085$ and $x_2 = 1.314$. For the right ear from $x_1 = 1.314$ to $x_2 = 1.825$, the inner limits are $y_1(x)$ and $y_2(x)$. The intersection curve for the ellipsoid lies completely above the ellipsoid's equator. The inner limits are $y_1(x)$ and $y_2(x)$. The outer limits are the real domain endpoints -1.160 and 1.825.

As illustrated by this example, an inconvenience with our approach is that the number of integrations required can differ depending on the radius of the sphere. For instance, if sphere's radius is increased to $r = 5$, six integrations are required to obtain the surface areas of 21.9 for the sphere and 64.0 for the ellipsoid. (1) in §7 contains the complete details of the solutions for these and other radii.

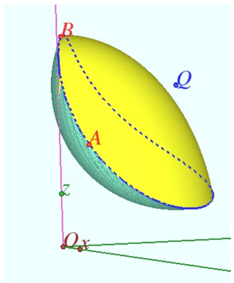


Figure 2: Enclosed Quadric Regions for Example 2

4.3 Example 3

For this rather interesting example, $(a, b, c) = (2, 3, 7.4)$, $(h, k, l) = (3.1, 3.35, 2.35)$, and $r = 7$. The spherical and ellipsoidal surface areas are 32.4 and 118.9, respectively. These surfaces are depicted in Figure 7. Figure 8 depicts the rather unusual curve of intersection of the sphere and ellipsoid. Portions of all four branches are required to determine the curve. Figure 9 depicts the projection of the curve onto the xy-plane and the actual three-dimensional curve. Figure 10 shows the projection of the portion of the curve above $z = 0$ onto the plane $z = 0$. Figure 11 shows the projections of the upper and lower halves of the curve onto the plane $z = l$. Inspection of the colored projected branches in these figures shows that a total of twenty

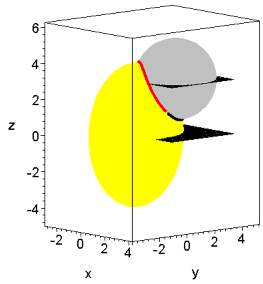


Figure 3: Curve and Surfaces for Example 2

(a) (b)

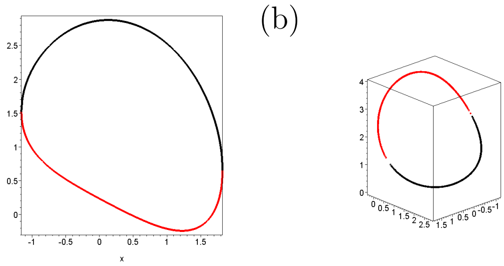


Figure 4: (a) Projected Intersection Curve (b) 3d Curve for Example 2

integrals (corresponding to color changes in either the upper or lower bounding curves) are required to evaluate the relevant ellipsoidal and spherical surface area integrals.

Notice that the intersection curve nearly self-intersects. For slightly larger values of r this indeed happens. After self-intersection occurs for slightly larger radii the curve breaks into two simple closed loops with the second loop shrinking to a single point as the radius increases. Smaller radii yield simple closed curves somewhat resembling those for Example 2.

5 Further Comments

The two-dimensional goat problem requires the calculation of areas bounded by either the circle or the ellipse and a slanted line (the line connecting the points of intersection of the circle and ellipse). Determining such areas is considered in [6]. The basic idea is that by using a rotation of axes we can find the areas with respect to a slanted line.

A similar approach can be used in our three-dimensional case. That is, the problem can be formulated to find the surface areas bounded by the quadrics and a slanted plane. Using projections onto other sensibly chosen planes rather than projecting onto the equator planes as we do can simplify the solution for certain special cases. This question is the subject of further investigation.

With our approach, determining the curve of intersection of the sphere and ellipsoid is quite

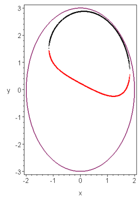


Figure 5: Ellipsoidal Projection: $z \geq 0$ for Example 2

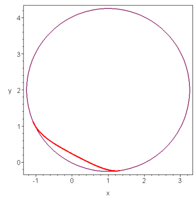
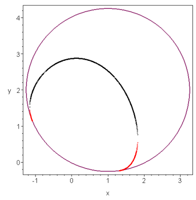
(a)  (b) 

Figure 6: Spherical Projections: (a) $z \geq l$ (b) $z \leq l$ for Example 2

involved. In addition, several integrations are sometimes required to find the relevant surface areas. The situation can be improved in a natural way by using parametric coordinates. This question is addressed in [5].

6 Summary

In this paper we considered the task of finding the curve of intersection of an ellipsoid and a sphere and that of finding the surface area of the portion of each quadric that is inside the other quadric. We described a solution procedure that can be used to accomplish each task. We presented examples that illustrate use of the procedure. We provided a rather general Maple worksheet that implements the method described. A parametric approach for finding the curve of intersection and the necessary surface areas is discussed in [5].

7 Supplemental Electronic Materials

Several Maple worksheets are available to perform the calculations described in this paper and to verify the given results. These worksheets include:

1. Sphere_Ellipsoid_Intersection.mws, the core worksheet to verify the results in this paper
2. Maple_procs.txt, a text file containing auxiliary procedures read and used by the other worksheets
3. On_xaxis.mws, a simplified worksheet for use if the center of the sphere lies on the x-axis
4. On_yaxis.mws, a simplified worksheet for use if the center of the sphere lies on the y-axis
5. Sphere_Sphere_Intersection.mws, a simplified worksheet for the intersection of two spheres

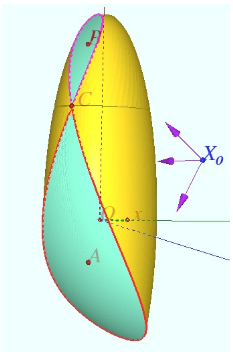


Figure 7: Enclosed Quadric Regions for Example 3

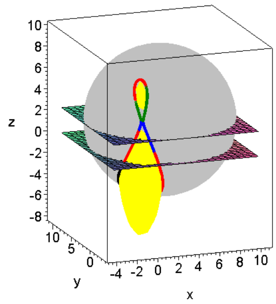


Figure 8: Curve and Surfaces for Example 3

(a) (b)

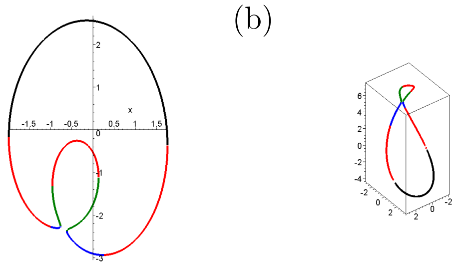


Figure 9: (a) Projected Intersection Curve (b) 3d Curve for Example 3

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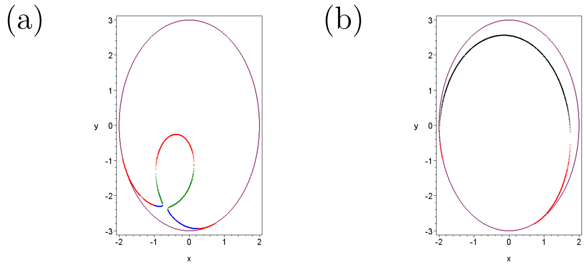

Figure 10: Ellipsoidal Projections: (a) $z \geq 0$ (b) $z \leq 0$ for Example 2

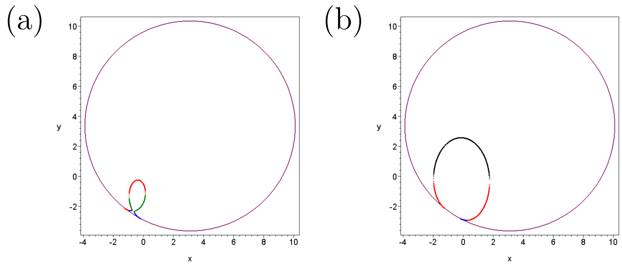

Figure 11: Spherical Projections: (a) $z \geq l$ (b) $z \leq l$ for Example 2

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